MODIFIED STEFAN PROBLEM*

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In analytical studies of solidification, one usually prescribes the shape of the growing solid and aims to determine the velocity of growth as a function of the various pertinent parameters. The present study assumes the velocity of growth and aims to determine the temperature of the growing surface, for the case of simple geometry. The former class of problems is appropriate for study of nucleus growth, and the latter for study of dendrite growth.

I. Introduction

Stefan's problem requires solution of the heat conduction equation in two phases, solid and liquid, separated by a moving interface on which heat is evolved proportional to the difference in temperature gradients in the two media, the interface being assumed isothermal. Of principal interest is the speed of motion of the interface. It was shown in the last century by F. Neumann and J. Stefan (see, e. g., Carslaw-Jaeger [1]) that for the one-dimensional case of plane front motion the relation between interface position R and time t is

$$R = 2\beta\sqrt{\kappa t}\,. \tag{1}$$

(κ is the thermal diffusivity of the liquid.) The liquid, of original temperature $T_\infty < T_f$ (T_f is the freezing temperature), extends initially from $x = 0$ to $x = \infty$ to the right; the solid of original temperature T_f extends from $x = 0$ to $x = -\infty$ on the left.** The "freezing parameter" β is a function of the dimensionless undercooling $(T_f - T_\infty)/\lambda$ where λ is the heat of fusion. It was subsequently shown by Ivantsov [3], Zener [4], and Frank [5] that spherical and cylindrical nuclei also obey the parabolic law of growth (1). The shape-preserving square root growth law (1) was subsequently shown by Ham [6], and again by Horvay-Cahn [7] to hold also for the general ellipsoid.*** In the latter case, R denotes an "equivalent radius." The dimensionless quantity $\beta = R/2\sqrt{\kappa t}$ is denoted in [7] by $\Omega^{1/2}$, and is referred to, following Frank, as "reduced radius."

A problem of interest equal to the original Stefan problem is the case where, in place of assuming an isothermal interface temperature and seeking to determine the speed of motion of the interface, one prescribes the motion of the interface and seeks to determine the temperature field in the two media; in particular, one wants to determine the temperature on the interface. From the preceding discussion it is clear that, if

$$V = \dot{R} = \beta\sqrt{\kappa/t} \tag{2}$$

is the prescribed velocity of motion, then the interface temperature is (spacewise) isothermal and (timewise) constant for planar, cylindrical, spherical, and, in fact, for general ellipsoidal growth. Of greatest interest to metallurgists is the case where the freezing front moves forward (say, in the positive x direction) at constant speed V; this is the case of form-preserving or "dendritic" growth.**** It was first shown by Ivantsov [3] that a paraboloid of revolution with an isothermal constant temperature interface grows at constant velocity and, conversely, if the paraboloid grows at constant velocity, then the interface temperature is isothermal and constant. This solution was subsequently extended by Horvay-Cahn [7] to arbitrary paraboloids.

There are several avenues of progress beyond the ellipsoid and paraboloid problems one may seek to follow. One may ask, for instance: State all shape- (or form-) preserving surfaces that are isothermal surfaces of constant temperature and represent a freezing front moving (A) as square root function of time, (B) as linear function of time. (C) Are there

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**In this report we are concerned with the case where the liquid is undercooled, so that the latent heat evolved is absorbed by the liquid. In many other analyses one presupposes $T_\infty \geq T_f$, $T_{-\infty} < T_f$ (see, e. g., [1], or Horvay-Henzel [2]); then the latent heat evolved is conducted into the solid.

***"Shape-preserving": the closed surface of the nucleus at time t is similar to its former form at t_0, but larger by factor $R(t)/R(t_0)$ in all dimensions.

****"Form-preserving": the open surface of the semi-infinite dendrite at time t is identical to the form at the previous time t_0, but is displaced by distance $V(t - t_0)$ in the direction of growth.

isothermal constant temperature shape-(or form-) preserving surfaces that grow with a law different from the $t^{1/2}$ or t^1 law?* It is surmised that ellipsoids and paraboloids exhaust classes A and B, and also C, but no formal proof is available. (D) How do dumbbell-shaped (or other nonconvex) surfaces grow; are these surfaces stable, or do they have a tendency to split? (This question has been answered by Mullins-Sekerka [8] for infinitesimal departures from sphere and plane when the velocity of growth is very small.) (E) If a constant temperature shape-preserving surface grows with a $t^{1/2}$ (or some other) law, how does the temperature vary with time if a different growth law (say, a t^1 law) is imposed? (This question has been answered by Kreith-Romie [9] for planes and inward freezing spheres and cylinders, using the t^1 law.) (F) Investigate nonparaboloidal form-preserving growth, and determine the variation of temperature along the freezing surface. (G) Refine the foregoing models by taking into account the effect of (a) density change upon freezing, (b) dependence of freezing temperature on surface tension and fluid pressure, (c) anisotropic materials properties; (d) replace the assumption of a unique freezing temperature T_f (corresponding to pure substances) by the assumption that freezing occurs over a temperature range $T_{upper} \leftrightarrow T_{lower}$ (corresponding to alloys; in such a case one must consider also (e) redistribution of solute in the liquid solvent during the freezing process).

Horvay has considered question (A) in [10a] for planes, spheres, cylinders, paraboloids, and question (B) in [10b] for spheres.** Various aspects of and various approximations to questions (B) through (E) and also (F) have been considered by Tiller [14], Jackson [15], Ivantsov [16], Temkin [17], Mullins-Sekerka [8], Hamilton [18], and previous to them, in a less complete manner, by many others. (For discussion and references to earlier literature, see, e.g., Ruddle [19], Chalmers [20], Turnbull [21], and Veynik [22].)

The list of questions could be considerably extended by referring to problems related to work carried on in many countries pertaining to (H) specific engineering structures, such as casting of finite slabs, finite cylinders, and other finite geometries, subject to various boundary conditions imposed by commercial molds (as contrasted with the case where nuclei or dendrites grow in infinite baths). These investigations are carried out by experiments (see, e.g., Pellini and his school [23a], Westwater and his school [23b], [19] and bibliography, and many others); by boundary layer type calculations (Goodman [24], Veynik [25]); by difference equation methods to be used in conjunction with digital computers (e.g., Douglas-Gallie [26a], Murray-Landis [26b] and bibliography, Ruddle [19] and bibliography, and many others); by simplified analytical methods (on occasion these may be of the form of two simultaneous difference equations, but compact simplified formulas rather than digital programming is the aim) such as London-Seban [27], Landau [28], Adams [29] Horvay [30], Bankoff-Hamill [31], Citron [32]; by passive analog methods, e.g., Paschkis [33]; and by active analog methods, e.g., [19, 30], and Baxter [34].

Parallel to, but independently of, the studies oriented toward the practical aspects of the subject, exhaustive research is being carried on into related abstract mathematical problems (a few representative papers are listed among the references of [31b] and item [35] of our bibliography) discussing primarily existence of solutions, their uniqueness, and their construction (frequently "in principle" only, not in practice).

In the present report and its sequels [36a, b] we shall be concerned with the constant velocity dendrite problem (F). The problem of form-preserving growth, with speed of growth prescribed, interface temperature unknown, is (from a mathematical point of view) intrinsically more tractable than the nucleus problem; where interface temperature is prescribed, speed of growth is unknown. In fact, this observation has frequently led to efforts to solve the nucleus problem in an appropriately reformulated way, as a dendrite problem, by means of integral equations; see, e.g., Lightfoot [37a], Selig [37b], Boley [37c], and others [35]. But there is a compensating difficulty. Whereas the nucleus problem as usually considered deals solely ([10b] is an exception) with an exterior heat conduction problem (the temperature needs to be determined only in the liquid because throughout the solid the temperature is a constant, T_f), the dendrite problem requires solution also of an interior heat conduction problem (since, except for the paraboloid, the interface is nonisothermal); the two solutions are to be matched at the interface. The situation simplifies greatly if the material properties of the frozen phase may be assumed equal to those of the liquid phase, because then the latent heat generation may be handled by the method of sources, as developed by Rosenthal [38].*** Subject to this assumption we shall determine in what follows some particulars of the temperature field about a growing dendrite, which, in Section IV, is of the form of a two-dimensional slab with an elliptical cap and in Section V is of the form of a three-dimensional cylinder with a spheroidal cap; this is known as Fisher's problem [41].

*We use the simple expression "the surface grows" to imply the growth of the nucleus or dendrite enclosed by it.

**Previously, Scriven [11] has considered the growth of a spherical nucleus into a denser melt, but he accounted only for the effect of fluid velocity on temperature, and disregarded the more exciting topic, the effect of fluid velocity on fluid pressure. The related problem of bubble formation (cavitation, boiling) has been extensively treated; see, e.g., Plesset [12] and its exhaustive bibliography, as well as Zwick [13a], Forster-Zuber [13b], etc.

***The method of sources is also used by Forster [39], Forster-Zuber [13b], and Yang-Clark [40] in somewhat related problems.

II. The Source Solution

Let a plane source parallel to the $\overline{YZ}$ plane, of intensity q" (Btu/ft^2-hr), move in the $\overline{X}$ direction at constant velocity V with respect to the fixed frame $\overline{XYZ}$ in a conducting medium of constant material properties, k = conductivity (Btu/ft-hr-deg); c = specific heat (Btu/lb-deg), γ = weight density (lb/ft^3); λ = heat of fusion (Btu/ft^3). Then the one-dimensional heat conduction equation

$$\partial T/\partial \bar{t} = \kappa \partial^2 T/\partial \overline{X}^2 \tag{3}$$

may be converted by the change of variables

$$X = \overline{X} - V\bar{t}, \quad Y = \overline{Y}, \quad Z = \overline{Z}, \quad t = \bar{t} \tag{4}$$

to one referred to the frame ZYZ moving with the plane front YZ, and the equation, for steady state, becomes

$$V\partial T/\partial X + \kappa \partial^2 T/\partial X^2 = 0\,. \tag{5}$$

This equation, subject to the boundary conditions

$$X = +\infty: \ T = 0; \quad X = 0: \ -k\,\partial T/\partial X = q'' \tag{6}$$

is solved by

$$X \geqslant 0: \ T = \frac{q''}{c\gamma V}\exp(-VX/\kappa); \quad X \leqslant 0: \ T = \frac{q''}{c\gamma V}\,. \tag{7}$$

If instead of a plane source q", we consider a point source q(X', Y', Z') (Btu/hr) moving in the positive $\overline{X}$ direction in a three-dimensional medium (see Fig. 1), then Eqs. (3), (5) are replaced by

$$\partial T/\partial \bar{t} = \kappa \overline{\nabla}^2 T, \quad V\partial T/\partial X + \kappa\, \nabla^2 T = 0, \tag{8a, b}$$

where bar over ∇^2 means that differentiation is to be with respect to the barred coordinates. The solution for the boundary conditions

$$\begin{gathered} X = +\infty: \ T = 0; \ (X, Y, Z) = (X', Y', Z'): \\ -4\pi k L^2 \frac{\partial T}{\partial L} = q, \\ L^2 = (X - X')^2 + (Y - Y')^2 + (Z - Z')^2, \end{gathered} \tag{9}$$

is

$$T = \exp[-V(L + X - X')/2\kappa]\, q/4\pi k L. \tag{10}$$

By integration $\int_{-\infty}^{\infty} dZ$ we then obtain in a two-dimensional medium, or in a three-dimensional medium per unit Z height, with line source of intensity q' (Btu/ft-hr) traveling in the positive $\overline{X}$ direction, the solution to Eq. (8b):

$$T = \frac{q'}{2\pi k}\exp[-V(X - X')/2\kappa] \times K_0(LV/2\kappa), \quad L^2 = (X - X')^2 + (Y - Y')^2 \tag{11}$$

(∇^2 is now the two-dimensional Laplacian) subject to the boundary conditions

$$X = \infty: \ T = 0; \ (X, Y) = (X', Y'): \ -2\pi k L \frac{\partial T}{\partial L} = q'. \tag{12}$$

These results are derived in Rosenthal's paper [38], and, differently, also in Carslaw-Jaeger [1, p. 266.] Note that Carslaw-Jaeger assume the source to move in the negative $\overline{X}$ direction, so they write (using the present notation) X' − X where we write X − X'.

Assume now that a dendrite grows in the $\overline{X}$ direction. Then to each surface element dA (whose normal is at angle θ to X; see Fig. 1) there is attached a source of strength

$$dq(X', Y', Z') = \gamma\lambda V \cos\theta \, dA(X', Y', Z'), \tag{13}$$

and the resultant temperature at point (X, Y, Z), on introducing (13) into (10) and integrating, becomes

$$T(X, Y, Z) = \frac{\gamma\lambda V}{4\pi k} \int\int \exp[-V(L + X - X')/2\kappa)] \cdot \frac{\cos\theta}{L} dA. \tag{14}$$

When the surface is a surface of revolution, with equation

$$R = R(-X'), \qquad R^2 = Z'^2 + Y'^2, \tag{15}$$

then, noting

$$dA \cos\theta = Rd\varphi \, dXdR/d(-X') \tag{16}$$

(φ is the latitude angle of dA with respect to the XY plane), Eq. (14) reduces to

$$T(X, Y, Z) = \frac{\gamma\lambda V}{4\pi k} \int_{-\infty}^{0} dX' \int_{0}^{2\pi} \frac{R}{L} \frac{dR}{d(-X')} \exp[-V(L + X - X')/2\kappa] \, d\varphi, \tag{17a}$$

$$L^2 = (X - X')^2 + (Y - R\cos\varphi)^2 + (Z - R\sin\varphi)^2. \tag{17b}$$

This formula was previously given by Ivantsov [16].

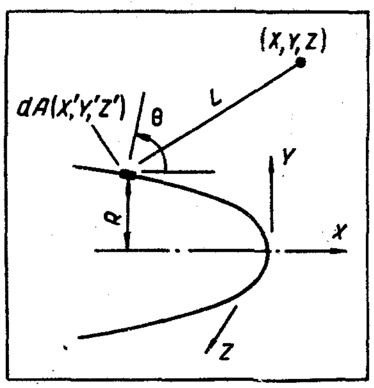

Fig. 1. The growing dendrite. $\bar{X}$, $\bar{Y}$, $\bar{Z}$ is a fixed coordinate system; the coordinate system X, Y, Z is attached to the traveling dendrite. (The location of the origin of the X, Y, Z system is selected at convenience. In Fig. 1 it is at the dendrite tip; in Fig. 3 it is at the ellipse center.)

For a cylindrical dendrite, symmetrical with respect to the ZX plane, having generators parallel to the Z axis (and a longitudinal section somewhat like the one in Fig. 1), the line source solution, with

$$dq'(X', Y') = \gamma\lambda V \cos\theta \, ds(X', Y'), \tag{18}$$

applies (ds is the line element), and

$$T(X, Y) = (\gamma\lambda V/2\pi k) \int_{0}^{\infty} \exp[-V(X - X')/2\kappa] [K_0(VL_+/2\kappa) + K_0(VL_-/2\kappa)] \, dY, \tag{19a}$$

$$L_{\pm}^2 = (X - X')^2 + (Y \mp Y')^2, \tag{19b}$$

$$-X' = -X'(Y') \tag{19c}$$

is the expression of the temperature distribution. Equation (19c) is the equation of the profile.

For the plane front dendrite one obtains from (7) on substituting

$$q'' = \gamma\lambda V \tag{20}$$

the result

1-dimensional:

$$u = \begin{cases} \exp(-2\mathrm{Pe}\, x), & x \geqslant 0, \\ 1, & x \leqslant 0, \end{cases} \tag{21a}$$

where

$$u = cT/\lambda,\ \mathrm{Pe} = Vd/2\kappa,\ r = R/d,\ l = L/d,\ x = X/d\ ,\ \text{etc.} \tag{22}$$

are dimensionless temperature, dimensionless speed of advance (Peclet number), and dimensionless distances. Here d is an arbitrary reference distance. For convenience we rewrite also Eqs. (19a) (17a) in dimensionless forms:

2-dimensional:
$$u = (\mathrm{Pe}/\pi) \int_0^\infty \exp\left[-\mathrm{Pe}(x-x')\right]\left[K_0(\mathrm{Pe}\, l_+) + K_0(\mathrm{Pe}\, l_-)\right] dy'; \tag{21b}$$

3-dimensional:
$$u = (\mathrm{Pe}/2\pi) \int_{-\infty}^{0} dx' \int_0^{2\pi} \frac{r}{l} \frac{dr}{d(-x')} \exp\left[-\mathrm{Pe}(l + x - x')\right] d\varphi. \tag{21c}$$

III. Fisher's Quasi-static Approximation

The dimensionless temperature about a sphere freezing at temperature $T(R) = T_f$ into an undercooled melt of original temperature $T(\infty) = T_\infty$ is (see, e. g., [7], Eq. (69a)),

$$u = u_f H(\omega)/H(\Omega), \quad H(\omega) = \int_\omega^\infty \mu^{-3/2} \exp(-\mu)\, d\mu, \tag{23a}$$

$$(T_f - T_\infty)\, c/\lambda \equiv u_f = \Omega^{3/2} \exp(\Omega) H(\Omega) = 2\Omega\left[1 - \sqrt{\pi\Omega} + 2\Omega + \ldots\right], \tag{23b}$$

$$\omega = \rho^2/2\kappa t,\ \beta^2 \equiv \Omega = R^2/2\kappa t. \tag{23c}$$

(ρ is the radial coordinate; R the radius to the freezing front.) Its gradient at the front is

$$\left.\frac{\partial u}{\partial \rho}\right|_R = -\frac{2\Omega}{\rho} = \tag{24a}$$

$$= -\frac{u_f}{R},\ \text{when}\ \Omega\ (\text{i.e.,}\ u_f) \ll 1, \tag{24b}$$

or, using dimensionless distances $r = \rho/d$, $R' = R/d$ (d = reference length)

$$\left.\frac{\partial u}{\partial r}\right|_{R'} = -\frac{u_f}{R'}\ \text{for}\ \Omega,\ u_f \ll 1. \tag{24c}$$

This, as was pointed out by Fisher 15 years ago (in an unpublished letter to B. Chalmers [41]; see also [8a]), agrees with the surface gradient obtained by solving the steady-state heat conduction equation of the sphere exterior (K = 2)

$$\frac{\partial^2 T}{\partial \rho^2} + \frac{K}{\rho}\frac{\partial T}{\partial \rho} = 0 \tag{25a}$$

for the boundary conditions

$$T(R) = T_f,\quad T(\infty) = T_\infty. \tag{25b}$$

One finds, indeed, that

$$T = (T_f - T_\infty)\, R/\rho + T_\infty,\ \left.\frac{\partial T}{\partial \rho}\right|_R = -\frac{T_f - T_\infty}{R} \tag{26a, b}$$

in accordance with (24b, c).

Using this observation as a starting point (namely, that for very small undercoolings the temperature field about a freezing sphere may be determined as for a static sphere, and hence, (26b) holds). Fisher set the heat released by a cylindrically tailed half sphere moving with velocity V

$$H_l' = \lambda\gamma\pi R^2 V, \tag{27a}$$

equal to the heat carried into the liquid by conduction

$$H_c' = [k\ (T_f - T_\infty)/R]\cdot 2\pi R^2, \tag{27b}$$

(he ignored the spherical nonsymmetry of the configuration), and obtained

$$u_f = (T_f - T_\infty)\,c/\lambda = RV/2\kappa = \mathrm{Pe}. \tag{28}$$

Thus he found that (at least near the tip of the sphere, where the present approximation is expected to be the most satisfactory) the freezing temperature u_f was Pe. (Fisher then went on to other considerations: by taking into account surface tension he determined a relation between R and V through the requirement (see also [17]) that the radius R of the dendrite cap should be that which maximizes V. However, we shall not get involved in these considerations.)

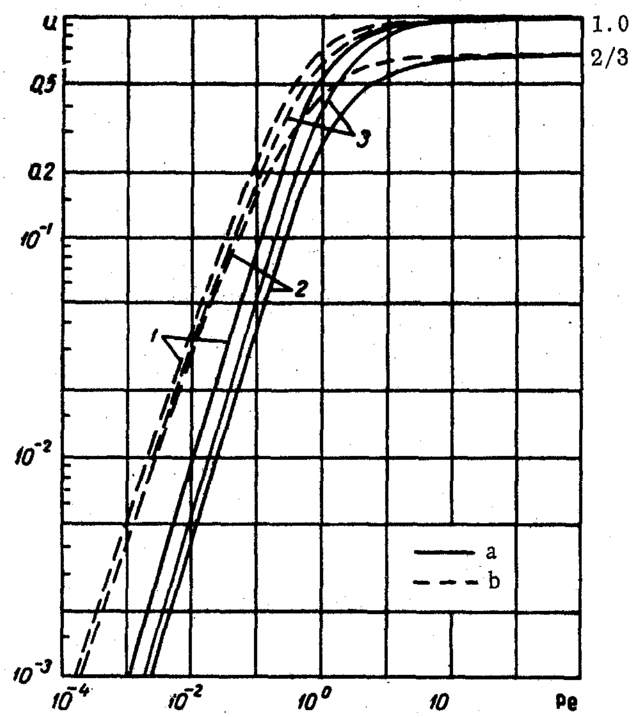

Fig. 2. Variation (with speed of growth, Pe = VR/2κ) of temperature at: 1) Tip (u_t); 2) cylinder hemisphere (slab-semicircle) junction (u_j); 3) origin (u_{0r}) of the cylindrical dendrite with hemispherical cap (slab dendrite with semicircular cap). a) Cylinder with hemispherical cap, b) slab with circular cap.

The present study stemmed from the desire to establish a more accurate picture of the temperature distribution along the moving dendrite. Indeed, we shall find in (75) that at the tip

$$u_t \simeq \begin{cases} 0.94\,\mathrm{Pe}, & \mathrm{Pe} \ll 1, \\ 0.518, & \mathrm{Pe} = 1, \\ 1 - (1/\mathrm{Pe}), & \mathrm{Pe} \gg 1, \end{cases} \tag{29a}$$

whereas at the juncture of sphere and cylinder [see (67)],

$$u_j \simeq \begin{cases} 0.42\,\mathrm{Pe}, & \mathrm{Pe} \ll 1, \\ 0.266, & \mathrm{Pe} = 1, \\ 2/3, & \mathrm{Pe} \gg 1, \end{cases} \tag{29b}$$

and very far from the tip

$$u_{far} \to 0 . \tag{29c}$$

So it is seen that the very simple Fisher approximation gives the right order of magnitude for u_t when $Pe \ll 1$. u_t, u_j, u_{0r} (the temperature at the sphere or circle center, Fig. 3) are plotted in Fig. 2 vs Pe for the cylinder with spherical cap and the slab with circular cap, on the basis of the formulas of Sections IV and V. Clearly, the spherically capped dendrite becomes hotter than its two-dimensional counterpart, because it involves more heat source area per unit volume.

On repeating the above considerations for the slab with a circular cap, (23) must be replaced (see (69b) of [7]) by

$$u = u_f H(\omega)/H(\Omega), \quad H(\omega) = \int_{\omega}^{\infty} \mu^{-1} \exp(-\mu)\, d\mu, \tag{30}$$

$$u_f = \Omega \exp(\Omega) H(\Omega) = \Omega \left(\ln \frac{1}{1.781\,\Omega} + \Omega \ln \frac{1}{1.781\,\Omega} + \Omega + \ldots \right).$$

Noting that for Ω, $u_f \ll 1$ the inversion of the u_f, Ω relation gives

$$\Omega \simeq u_f \left(\ln \frac{1}{1.781\, u_f} \right)^{-1}, \tag{31}$$

the Eqs. (24) become replaced by

$$\left. \frac{\partial u}{\partial r} \right|_{R'} = -\frac{2\Omega}{R'} = \frac{2u_f / \ln 1.781\, u_f}{R'}, \tag{32}$$

and (27a, b), (28) by

$$H_l' = 2R\lambda\gamma V, \quad H_c' = \frac{\lambda k}{c} \frac{2\Omega}{R} \pi R, \tag{33a, b}$$

$$u_f \left(\ln \frac{1}{1.781\, u_f} \right)^{-1} = \Omega = \frac{RV}{\kappa\pi} = \frac{2\mathrm{Pe}}{\pi}, \quad u_f = \frac{2\mathrm{Pe}}{\pi} \ln \frac{\pi/2}{1.781\,\mathrm{Pe}}. \tag{34a, b}$$

(34b) compares with the results established in Section IV:

$$u_t = \begin{cases} \dfrac{2\mathrm{Pe}}{\pi} \ln \dfrac{2}{1.781\,\mathrm{Pe}}, & \mathrm{Pe} \ll 1, \\ 0.709, & \mathrm{Pe} = 1, \\ 1-(1/2\,\mathrm{Pe}), & \mathrm{Pe} \gg 1, \end{cases} \tag{35a}$$

$$u_j = \begin{cases} \dfrac{2\mathrm{Pe}}{\pi} \ln \dfrac{1}{1.781\,\mathrm{Pe}}, & \mathrm{Pe} \ll 1, \\ 0.469, & \mathrm{Pe} = 1, \\ \dfrac{2}{3} \left(1 - \exp(-\mathrm{Pe})/\sqrt{\pi\,\mathrm{Pe}}\right), & \mathrm{Pe} \gg 1, \end{cases} \tag{35b}$$

$$u_f \to 0. \tag{35c}$$

It is noted that for $\mathrm{Pe} \ll 1$ the value (34b) is some mean of (35a), (35b). However, in the present instance the results (32), (34) cannot be obtained by a Fisher type analysis. The reason for this is that solution of (25a) for K = 1 (cylinder) gives

$$T = [T_f \ln(R^{\infty}/\rho) + T_{\infty} \ln(\rho/R)]/\ln(R^{\infty}/R), \tag{36a}$$

where the imposed boundary conditions are

$$T(R) = T_f,\ T(R^{\infty}) = T_{\infty},\ \infty > R^{\infty} > R, \tag{36b}$$

and the passage to $R^{\infty} \to \infty$ cannot be undertaken.

IV. The Slab with an Elliptical Cap

Equation (21b) reduces for a slab, 2R wide, and having an elliptical cap (see Fig. 3) to

$$u(x,\ y) = \frac{\mathrm{Pe}}{\pi a} \int_0^a \frac{x' \exp[-\mathrm{Pe}(x - x')]}{\sqrt{a^2 - x'^2}} \times [K_0(\mathrm{Pe}\, l_+) + K_0(\mathrm{Pe}\, l_-)]\, dx', \tag{37a}$$

$$u(0,\ y) = \frac{\mathrm{Pe}}{\pi} \int_0^1 \exp(\mathrm{Pe}\, a \sqrt{1 - y'^2})[K_0(\mathrm{Pe}\, l_+) + K_0(\mathrm{Pe}\, l_-)]\, dy', \tag{37b}$$

where

$$(x'^2/a^2) + y'^2 = 1 \tag{38a}$$

is the equation of the cap referred to R as unit length, and

$$\mathrm{Pe} = VR/2\kappa,\quad l_{\pm}^2 = (y \mp y')^2 + (x - x')^2. \tag{38b}$$

On denoting

$$\sigma = \mathrm{Pe}(a - x'),\quad t^2 = 2\mathrm{Pe}\,\sigma/a \tag{39}$$

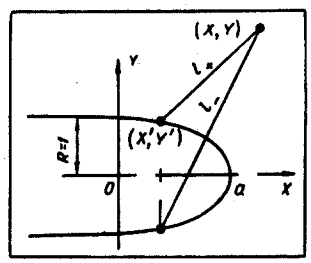

Fig. 3. Slab dendrite with elliptical cap.

(37a) assumes at the tip (x, y) = (a, 0) of the ellipse the expression

$$u_t = \frac{2}{\pi a} \int_0^{\mathrm{Pe}a} \frac{\mathrm{Pe}\, a - \sigma}{\sqrt{\sigma(2\mathrm{Pe}\, a - \sigma)}} \exp(-\sigma) K_0 \left\{\left[\frac{2\mathrm{Pe}}{a}\sigma + (1 - 1/a^2)\sigma^2\right]^{1/2}\right\} d\sigma = \tag{40a}$$

$$= \frac{2}{\pi} \int_0^{\mathrm{Pe}a} \exp(-\sigma) \sqrt{\frac{\mathrm{Pe}}{2a\sigma}} \left[1 - \frac{3}{4}\frac{\sigma}{\mathrm{Pe}\, a} - \frac{5}{32}\left(\frac{\sigma}{\mathrm{Pe}\, a}\right)^2 \dots\right] K_0 \left\{\sqrt{\frac{2\mathrm{Pe}\,\sigma}{a}}\left[1 + \right.\right.$$

$$\left.\left. + \frac{a^2 - \sigma}{4a\mathrm{Pe}}\sigma - \frac{(a^2 - 1)^2 \sigma^2}{32 a^2 \mathrm{Pe}^2} \dots\right]\right\} d\sigma =$$

$$= \frac{2}{\pi} \int_0^{\mathrm{Pe}\sqrt{2}} \exp(-at^2/2\mathrm{Pe}) \left(1 - \frac{3}{8}\frac{t^2}{\mathrm{Pe}^2} - \frac{5}{128}\frac{t^4}{\mathrm{Pe}^4} \dots\right) \times$$

$$\times K_0 \left[t + \frac{a^2 - 1}{8\mathrm{Pe}^2} t^3 - \frac{(a^2 - 1)^2}{128\mathrm{Pe}^4} t^5 \dots\right] dt = \tag{40b}$$

$$= \frac{2}{\pi}\left(\int_0^{\infty} - \int_{\mathrm{Pe}\sqrt{2}}^{\infty}\right) \exp(-at^2/2\mathrm{Pe}) \times \left\{\left[1 - \frac{3}{8}\frac{t^2}{\mathrm{Pe}^2} + \frac{(a^2 - 1)^2 t^6 - 10t^4}{256\,\mathrm{Pe}^4}\right] K_0(t) - \right.$$

$$\left. - \frac{a^2 - 1}{8\mathrm{Pe}^2}\left[t^3 - \frac{a^2 + 5}{16\mathrm{Pe}^2} t^5 \dots\right] K_1(t) + \left[\frac{(a^2 - 1)^2}{256\,\mathrm{Pe}^4} t^6 \dots\right] K_2(t) + \dots\right\} dt. \tag{40c}$$

For Pe ≫ 1 the integral $\int\limits_{\mathrm{Pe}\sqrt{2}}^{\infty}$ is exponentially small, and noting

$$\int\limits_0^\infty \exp(-\alpha t^2)\, t^n K_\nu(t)\, dt = 2^{n-1}\Gamma\left(\frac{n+\nu+1}{2}\right)\Gamma\left(\frac{n-\nu+1}{2}\right)\times$$
$$\times\left[1-\frac{(n+\nu+1)(n-\nu+1)}{1!}\alpha+\right. \qquad (41)$$
$$\left.+\frac{(n+\nu+1)(n+\nu+3)(n-\nu+1)(n-\nu+3)}{2!}\alpha^2-\dots\right],$$

which results from* [T_{II}, p. 132, No. 25] and [F_I, pp. 264, 278], (40c) leads to

$$u_t = 1-\frac{a}{2\mathrm{Pe}}+\frac{3}{4}\frac{a^2}{\mathrm{Pe}^2}-\frac{(15a^2+9)a}{8\mathrm{Pe}^3}+\frac{15}{16}\frac{(7a^2+12)a^2}{\mathrm{Pe}^4}-\dots \qquad (40d)$$

For Pe ≪ 1 one finds, on the other hand, to Pe^2 terms, the ascending series

$$u_t = \frac{2\mathrm{Pe}}{\pi}\int\limits_0^1 \frac{1-s}{\sqrt{s(2-s)}}[1-a\mathrm{Pe}\, s]\times\left\{\ln\frac{2}{\gamma\,\mathrm{Pe}}-\frac{1}{2}\ln[2s+(a^2-1)s^2]\right\}ds \simeq \qquad (42a)$$

$$\simeq \frac{2\mathrm{Pe}}{\pi}\left[1-\mathrm{Pe}a\left(1-\frac{\pi}{4}\right)\right]\ln\frac{2}{\gamma\,\mathrm{Pe}\sqrt{1+a^2}}+\frac{\mathrm{Pe}}{\pi}\left[1+\frac{\pi}{2}+(a^2-1)C_1(a)\right]-$$
$$-\frac{\mathrm{Pe}^2 a}{2\pi}\left[\frac{3\pi}{4}+1-2.94+(a^2-1)C_2(a)\right], \qquad (42b)$$

where

$$\ln\gamma = \ln 1.781 = 0.5772 = \text{Euler-Mascheroni constant}, \qquad (43a)$$

$$C_1(a) = \int\limits_0^1 \frac{\sqrt{s(2-s)}}{2+(a^2-1)s}\,ds, \qquad (43b)$$

$$C_2(a) = \int\limits_0^1 \frac{(1+s)\sqrt{s(2-s)}-\arccos s}{2+(a^2-1)s}\,ds, \qquad (43c)$$

and we have utilized the formula obtained by numerical integration

$$\int\limits_0^1 \frac{\arccos u}{1-u}\,du = 2.94. \qquad (43d)$$

Formulas (42b) and (40d) are almost usable up to a = Pe = 1. For a = Pe = 1 the former gives 0.64 (too small because the next term in (42b) is positive), the latter suggests a value between 0.7 and 0.8. A numerical integration by formula (50) will show that

*References [42a, 42b] are denoted by [T] (Tables) and [F] (Functions), respectively.

$$a = \mathrm{Pe} = 1: \quad u_t = 0.709. \tag{44}$$

For the origin (x, y) = (0, 0) formula (37a) reduces to

$$u_{0r} = \frac{2\mathrm{Pe}}{\pi a} \int_0^a \frac{x' \exp(\mathrm{Pe}\, x')}{\sqrt{a^2 - x'^2}} K_0\left\{\mathrm{Pe}\sqrt{1 + x'^2(1 - 1/a^2)}\right\} dx'. \tag{45}$$

Noting [T_I, p. 138, no. 12; p. 136, no. 28] (the latter is valid also for Re Pe < 0), it leads for the case of the circle, $a = 1$, to the closed formula*

$$u_{0r} = \mathrm{Pe}\, K_0(\mathrm{Pe})\left[\frac{2}{\pi} + I_1(\mathrm{Pe}) + L_1'(\mathrm{Pe})\right]. \tag{46}$$

For $a \neq 1$, $\mathrm{Pe} \ll 1$, (45) gives, to Pe^2 terms

$$\begin{matrix} a > 1: \\ a < 1: \end{matrix} \; u_{0r} = \frac{2\mathrm{Pe}}{\pi} \ln \frac{2e}{\gamma\, \mathrm{Pe}} - \frac{a\,\mathrm{Pe}}{\pi} \times \begin{Bmatrix} (a^2-1)^{-1/2} \ln[(a + \sqrt{a^2-1})/(a - \sqrt{a^2-1})] \\ 2(1-a^2)^{-1/2} \arccos a \end{Bmatrix} -$$

$$- \frac{a\,\mathrm{Pe}^2}{2}\left[C_3(a) - \ln \frac{2}{\gamma\,\mathrm{Pe}}\right], \tag{47}$$

$$C_3(a) = \frac{1}{\pi} \int_0^1 \sqrt{\frac{1-u}{u}} \ln\{a^2 + (1-a^2)u\}\, du.$$

For $a \neq 1$, $\mathrm{Pe} \gg 1$ a (not very pleasant) numerical evaluation of (45) is recommended.

For a slab with a circular cap, $a = 1$, (37a) leads in polar coordinates

$$x' = \cos\theta, \quad y' = \sin\theta \tag{48a}$$

for points along the arc

$$x = \cos\Theta, \quad y = \sin\Theta \tag{48b}$$

to the expression

$$u = \frac{\mathrm{Pe}}{\pi} \exp(-\mathrm{Pe}\cos\Theta) \int_0^{\pi/2} \exp(\mathrm{Pe}\cos\theta)\left[K_0\left(2\mathrm{Pe}\sin\frac{\theta+\Theta}{2}\right) + \right.$$

$$\left. + K_0\left(\left|2\,\mathrm{Pe}\sin\frac{\theta-\Theta}{2}\right|\right)\right] \cos\theta\, d\theta. \tag{49}$$

At the tip, $\Theta = 0$, this becomes (for arbitrary Pe)

$$u_t = u(1{,}0) = \frac{2\mathrm{Pe}}{\pi} \exp(-\mathrm{Pe}) \times \int_0^{\pi/2} \exp(\mathrm{Pe}\cos\theta) K_0\left(2\,\mathrm{Pe}\sin\frac{\theta}{2}\right) \cos\theta\, d\theta =$$

*L_k' (Pe) are Struve's functions. For $\mathrm{Pe} \gg 1$:

$$L_0'(\mathrm{Pe}) = I_0(\mathrm{Pe}) - \frac{2}{\pi\,\mathrm{Pe}}, \quad L_1'(\mathrm{Pe}) = I_1(\mathrm{Pe}) - \frac{2}{\pi}\left(1 - \frac{1}{\mathrm{Pe}^2}\right).$$

$$= \frac{4\mathrm{Pe}}{\pi} \int_0^{1/\sqrt{2}} \frac{1 - 2\lambda^2}{\sqrt{1-\lambda^2}} \exp(-2\,\mathrm{Pe}\,\lambda^2)\, K_0(2\,\mathrm{Pe}\,\lambda)\, d\lambda. \quad (50)$$

This expression lends itself to convenient numerical integration.

For the juncture (x, y) = (0, 1) of general ellipse and slab (37b) reduces to

$$u_j = \frac{\mathrm{Pe}}{\pi} \int_0^1 \exp\left[(\mathrm{Pe}\,a(1-y'^2)^{1/2}\right] \times (K_0\{\mathrm{Pe}[(1-y')^2 + a^2(1-y'^2)]^{1/2}\} +$$

$$+ K_0\{\mathrm{Pe}[(1+y')^2 + a^2(1-y'^2)]^{1/2}\})\, dy', \quad (51)$$

and we obtain in a one-term approximation

$$\mathrm{Pe} \gg 1:\ u_j \simeq \sqrt{\frac{\mathrm{Pe}}{2\pi}} \int_0^1 \exp[\mathrm{Pe}\,a(1-y'^2)^{1/2}] \times$$

$$\times \left\{ \frac{\exp\{-\mathrm{Pe}[(1-y')^2 + a^2(1-y'^2)]^{1/2}\}}{[(1-y')^2 + a^2(1-y'^2)]^{1/4}} + \frac{\exp\{-\mathrm{Pe}[(1+y')^2 + a^2(1-y'^2)]^{1/2}\}}{[(1+y')^2 + a^2(1-y'^2)]^{1/4}} \right\} dy'. \quad (52)$$

This formula may be specialized to

$$a^2 \ll 1,\ (a\,\mathrm{Pe})^2 \gg 1:\ u_j \simeq \sqrt{\frac{\mathrm{Pe}}{2\pi}} \exp(-\mathrm{Pe}\,a^2) \times$$

$$\times \int_0^2 t^{-1/2} \exp[-\mathrm{Pe}(1 - a^2/2)t + \mathrm{Pe}\,a(2t - t^2)^{1/2}]\, dt, \quad (53a)$$

$$a^2 = 1:\quad u_j \simeq \frac{2}{3} \operatorname{erf} \sqrt{\mathrm{Pe}}, \quad (53b)$$

$$a^2 \gg 1:\quad u_j \simeq \sqrt{\mathrm{Pe}/2\pi a} \int_0^2 (2t - t^2)^{-1/4} \times$$

$$\times \exp[-\mathrm{Pe}\, t^{3/2}/2a(2-t)^{1/2}]\, dt. \quad (53c)$$

For Pe ≪ 1 an approximation to Pe^2 terms gives

$$u_j = \frac{\mathrm{Pe}}{\pi} \int_0^1 (1 + \mathrm{Pe}\,a\sqrt{1-y'^2}) \Big\{ 2 \ln \frac{2}{\gamma\,\mathrm{Pe}} - \frac{1}{2} \ln [y'^4 (1-a^2)^2 -$$

$$- 2y'^2(1+a^4) + (1+a^2)^2] \Big\}\, dy', \quad (54)$$

which, on noting the definite integral

$$\int_0^1 (1-y^2)^{1/2} \ln(1-y^2)^{1/2}\, dy = \frac{\pi}{4} \ln \frac{\sqrt{e}}{2} \quad (55)$$

[this is obtained from* ($\mathrm{GH_{II}}$, p. 79, no. 52a) by simple substitution], leads to

*Reference 43 is denoted by GH.

$$a^2 \ll 1:\quad u_j = \frac{2\mathrm{Pe}}{\pi}\left[\left(1+\frac{\mathrm{Pe}\,a\,\pi}{4}\right)\ln\frac{2}{\gamma\mathrm{Pe}} + \ln\frac{e}{2} + \frac{a\,\mathrm{Pe}\,\pi}{4}\ln\frac{2}{\sqrt{e}}\right], \tag{56a}$$

$$a = 1:\quad u_j = \frac{2\,\mathrm{Pe}}{\pi}\left[\left(1+\frac{\mathrm{Pe}\,\pi}{4}\right)\ln\frac{\sqrt{2}}{\gamma\,\mathrm{Pe}} + \frac{1}{2}\ln\frac{e}{2} + \frac{\mathrm{Pe}\,\pi}{8}\ln\frac{2}{\sqrt{e}}\right], \tag{56b}$$

$$a^2 \gg 1,\ (a\,\mathrm{Pe})^2 \ll 1:\quad u_j = \frac{2\,\mathrm{Pe}}{\pi}\left[\left(1+\frac{\mathrm{Pe}\,a\,\pi}{4}\right)\times\ln\frac{2}{a\gamma\,\mathrm{Pe}} + \ln\frac{e}{2} + \frac{a\,\mathrm{Pe}\,\pi}{4}\ln\frac{2}{\sqrt{e}}\right]. \tag{56c}$$

Finally, when $|y| \lesssim 1$, $|x| \gg 1$, one may write

$$l_\pm = |x|\{(1-x'/x)^2 + (y'\mp y)^2/x^2\}^{1/2} \simeq |x|\left[1-\frac{x'}{x}+\frac{1}{2}\left(\frac{y'\mp y}{x}\right)^2\right], \tag{57}$$

and (37a) becomes for Pe $\gg$ 1

$$u_{\mathrm{far}} = \frac{1}{a}\sqrt{\frac{\mathrm{Pe}}{2\pi}}\int_0^a \frac{x'}{\sqrt{a^2-x'^2}}\times\Big\{\exp[-\mathrm{Pe}\,(x-x'+l_+)]\,[\,l_+^{-1/2} - \frac{1}{8\,\mathrm{Pe}}\,l_+^{-3/2} + \dots] +$$

$$+\exp[-\mathrm{Pe}\,(x-x'+l_-)]\,[\,l_-^{-1/2} - \frac{1}{8\,\mathrm{Pe}}\,l_-^{-3/2} + \dots]\Big\}\,dx, \tag{58}$$

from which there results

Pe $\gg$ 1:

$$y=0,\quad x\gg 1:\ u_{\mathrm{far}} \simeq \sqrt{\frac{\mathrm{Pe}\,\pi}{2x}}\exp(-2\,\mathrm{Pe}\,x)\times\Big\{\left(1-\frac{1}{8\,\mathrm{Pe}\,x}\right)\left[\frac{2}{\pi} + L_1'(2a\,\mathrm{Pe}) +\right.$$

$$\left.+\,I_1(2a\,\mathrm{Pe})\right] + \frac{a}{2x}\left[L_0'(2a\,\mathrm{Pe}) + I_0\big(2a\,\mathrm{Pe}\big) - \frac{L_1'(2a\,\mathrm{Pe}) + I_1(2a\,\mathrm{Pe})}{2a\,\mathrm{Pe}}\right]\Big\},\quad y=0,\ x\ll -1: \tag{59a}$$

$$u_{\mathrm{far}} \simeq \left(1-\frac{a}{2|x|}+\frac{2a|x|-1}{8\,\mathrm{Pe}\,|x|}\right)\times \mathrm{erf}\left(\sqrt{\frac{\mathrm{Pe}}{2|x|}}\right) - \frac{a}{\sqrt{8\pi\,\mathrm{Pe}\,|x|}}\exp(-\mathrm{Pe}/2|x|), \tag{59b}$$

$$y=1,\quad x\ll -1:\ u_{\mathrm{far}} \simeq \frac{1}{2}\left(1-\frac{1}{8\,\mathrm{Pe}\,|x|}\right)\times\mathrm{erf}\left(\sqrt{\frac{2\,\mathrm{Pe}}{|x|}}\right) - a\sqrt{\frac{2\,\mathrm{Pe}}{\pi|x|^3}}\ \times$$

$$\times\int_0^1 [\exp(-2\,\mathrm{Pe}\,v^2/|x|)]\sqrt{v(1-v)}\,dv. \tag{59c}$$

For Pe $\ll$ 1, $|px| \ll 1$ we may write

$$u = \frac{\mathrm{Pe}}{\pi a}\exp(-\mathrm{Pe}\,x)\int_0^a \frac{x'(1+\mathrm{Pe}\,x')}{\sqrt{a^2-x'^2}}\left(2\ln\frac{2}{\gamma\,\mathrm{Pe}} - \ln l_+ l_-\right)dx', \tag{60}$$

from which there results

Pe $\ll$ 1, $|px| \ll 1$:

$$y=0,\ x\gg 1:\ u_{far}=\frac{2Pe}{\pi}\exp(-Pe\,x)\times\left[\left(1+\frac{Pe\,a\,\pi}{4}\right)\ln\frac{2}{\gamma\,Pe\,x}+\frac{a}{4x}\times\right.$$

$$\left.\times\left(\pi+\frac{8}{3}Pe\,a\right)+\frac{1}{2x^2}\left(\frac{-1+2a^2}{3}+\frac{-1+3a^2}{16}Pe\,a\,\pi\right)\right], \tag{61a}$$

$$y=0,\ x\ll -1:\ u_{far}=\frac{2\,Pe}{\pi}\exp(Pe\,|x|)\left[\left(1+\frac{Pe\,a\,\pi}{4}\right)\ln\frac{2}{\gamma\,Pe\,|x|}-\right.$$

$$\left.-\frac{a}{4\,|x|}\left(\pi+\frac{8}{3}Pe\,a\right)+\frac{1}{2x^2}\left(\frac{-1+2a^2}{3}+\frac{-1+3a^2}{16}Pe\,a\,\pi\right)\right], \tag{61b}$$

$$y=1,\ x\ll -1:\ u_{far}=\frac{2\,Pe}{\pi}\exp(Pe\,|x|)\left[\left(1+\frac{Pe\,a\,\pi}{4}\right)\ln\frac{2}{\gamma\,Pe\,|x|}-\right.$$

$$\left.-\frac{a}{4\,|x|}\left(\pi+\frac{8}{3}Pe\,a\right)+\frac{1}{2x^2}\left(\frac{-4+2a^2}{3}+\frac{-5+3a^2}{16}Pe\,a\,\pi\right)\right]. \tag{61c}$$

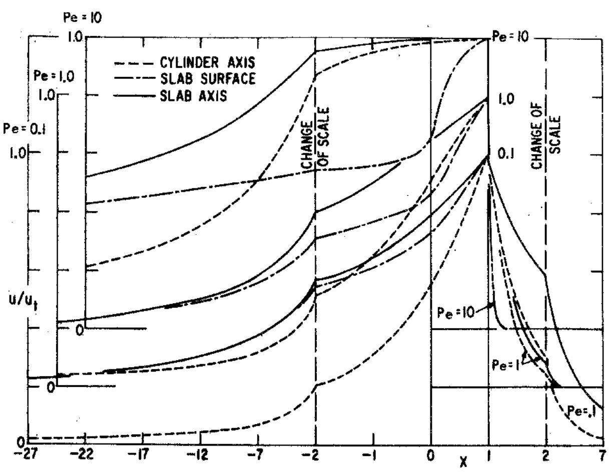

Fig. 4. Temperature along the cylinder axis, slab surface, slab axis, at values Pe = 10, 1, 0.1 of the Peclet number.

In Fig. 4 we plot for the slab with circular cap the temperature along the surface and axis (the latter also for the cylinder with spherical cap, by the methods of the next section) for $Pe = 10^{-1}$, 1, 10. In Fig. 5 the tip temperature is plotted for the slab with elliptical cap and for the cylinder with spheroidal cap for various axis ratios $a : 1$, when (a) $Pe \ll 1$, (b) $Pe \gg 1$. In Fig. 6 we plot similarly u_{or}/u_t for slab with elliptical cap and cylinder with shperoidal cap. The curve of u_j/u_t is drawn in the same figure for the slab with elliptical cap for the case $Pe = 10^{-3}$ only, because the stringent restrictions in (53), (56) do not permit us to draw the curves for $Pe = 10^{-1}$, $Pe = 10$. Only the $a = 0$, 1 points are indicated (by crosses) for these other cases, where direct use was made of (51).

V. The Cylinder with the Spheroidal Cap

Equation (21c) reduces for a cylinder of radius R = 1, having a spheroidal cap

$$x'^2/a^2+r^2=1 \quad (r^2=y'^2+z'^2) \tag{62}$$

to

$$u=\frac{Pe}{2\pi a^2}\int_{-\pi}^{\pi}d\varphi\int_0^a\frac{x'}{l}\exp\left[-Pe\,(l+x-x')\right]dx'. \tag{63}$$

Figure 7 illustrates the case where the point (x, y, z) is on the surface of the spheroid. Then

$$l^2=(x-x')^2+\left(\sqrt{1-x^2/a^2}-r\cos\varphi\right)^2+r^2\sin^2\varphi; \tag{64a}$$

while, when (x, y, z) is on the cylindrical portion of the surface, at the junction of cap and cylinder, and on the x axis, then

$$l^2=(x-x')^2+(1-r)^2+4r\sin^2\vartheta,\ \vartheta=\varphi/2, \tag{64b}$$

$$l^2=1+a^2-2r\cos\varphi+r^2(1-a^2), \tag{64c}$$

$$l^2 = (x - x')^2 + r^2. \tag{64d}$$

respectively.

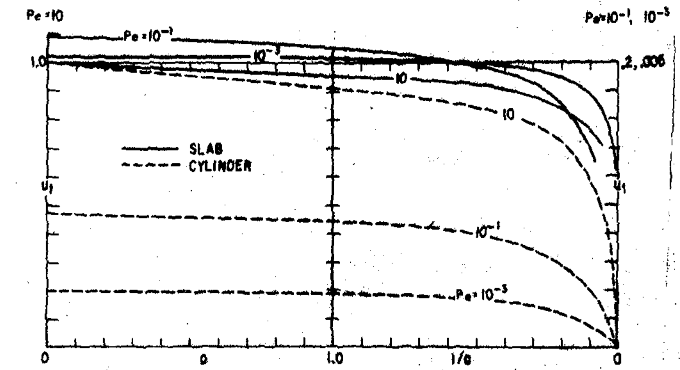

Fig. 5. Dependence of dendrite tip temperature on axis ratio a: 1 (a is in travel direction) of spheriod (ellipse), for Pe = 10, 10^{-1}, 10^{-3}.

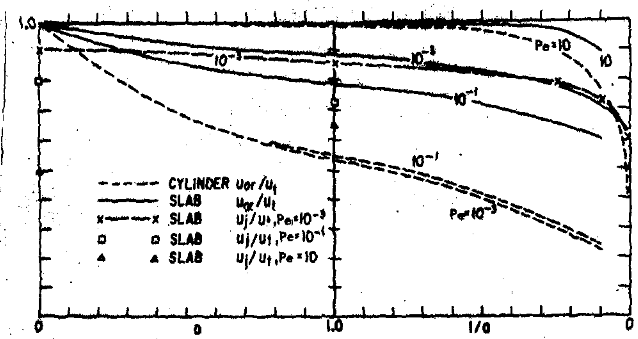

Fig. 6. Dependence of u_{or}/u_t, u_j/u_t on axis ratio, for Pe = 10, 10^{-1}, 10^{-3}.

For the sake of simplicity we shall restrict ourselves mostly to axial points $(x, 0, 0)$ where (63) reduces to

$$u = \frac{\mathrm{Pe}}{a^2} \int_0^a \frac{x'}{[(x - x')^2 + r^2]^{1/2}} \times \exp(-\mathrm{Pe}\,\{x - x' + [(x - x')^2 + r^2]^{1/2}\})\,dx'. \tag{65}$$

The only nonaxial point we shall consider is the junction (0, 1, 0) of sphere (a = 1) and cylinder, for which (63), with

$$r = \cos\lambda \tag{66}$$

becomes

$$u = \frac{\mathrm{Pe}}{\pi} \int_0^\pi d\varphi \int_0^1 \frac{r}{l} \exp(-\mathrm{Pe}\,[l - (1 - r^2)^{1/2}])\,dr =$$

$$= \frac{\mathrm{Pe}}{\pi} \int_0^{\pi/2} \psi_{\mathrm{Pe}}(\lambda) \exp(\mathrm{Pe} \sin\lambda)\, \frac{\sin 2\lambda}{2\lambda}\, d\lambda, \tag{67a}$$

$$\psi_{\mathrm{Pe}}(\lambda) = \lambda \int_0^\pi \frac{\exp\{-\mathrm{Pe}\,[2(1 - \cos\lambda\cos\varphi)]^{1/2}\}}{[2(1 - \cos\lambda\cos\varphi)]^{1/2}}\, d\varphi, \tag{67b}$$

$\psi_{\mathrm{Pe}}(\lambda)$ is plotted in Fig. 8.

The special case $\psi_0(\lambda)$ may be expressed in terms of the tabulated complete elliptic integral of the first kind, $K(k')$ – see [44] formula 282.00:

$$\psi_0(\lambda) = 2\lambda \int_0^{\pi/2} [2 - 2\cos\lambda + 4\cos\lambda \sin^2\vartheta]^{-1/2}\, d\vartheta = \left(\frac{\lambda}{\sin\lambda/2}\right) \times$$

$$\times \int_0^{\pi/2} (1 + n^2 \sin^2\vartheta)^{-1/2}\, d\vartheta = \frac{\lambda}{\cos\lambda/2} K(k); \tag{68}$$

$$n^2 = 2\cos\lambda/(1 - \cos\lambda),\quad k^2 = \frac{n^2}{1 + n^2} = \frac{2\cos\lambda}{1 + \cos\lambda} = \frac{\cos\lambda}{\cos^2\lambda/2}.$$

Thus we obtain

$$\mathrm{Pe} \ll 1:\quad u_j \simeq \frac{2\,\mathrm{Pe}}{\pi} \int_0^{\pi/2} \sin\frac{\lambda}{2} \cos\lambda \times K\left(\frac{\sqrt{\cos\lambda}}{\cos\lambda/2}\right) d\lambda = 0.424\,\mathrm{Pe}. \tag{69}$$

For Pe ≫ 1, and denoting

$$z = [2(1 - \cos\lambda \cos\varphi)]^{1/2} \tag{70}$$

we may write

$$\psi_{\mathrm{Pe}}(\lambda) = \lambda \int\limits_{[2(1-\cos\lambda)]^{1/2}}^{[2(1+\cos\lambda)]^{1/2}} [\cos\lambda + 1 - z^2/2]^{-1/2} [\cos\lambda - 1 + z^2/2]^{-1/2} \times \exp(-\mathrm{Pe}\, z)\, dz$$

$$\simeq \frac{\lambda}{\sqrt{\cos\lambda}} \times \int\limits_{(2-2\cos\lambda)^{1/2}}^{\infty} [z^2 - 2(1-\cos\lambda)] \times \exp(-\mathrm{Pe}\, z)\, dz = \frac{\lambda}{\sqrt{\cos\lambda}} K_0(2\,\mathrm{Pe} \sin\lambda/2). \tag{71}$$

The center expression of ψ_{Pe} is obtained on noting that most of the integral is contributed by the vicinity of $z = (2-2\cos\lambda)^{1/2}$; hence the upper limit may be replaced by ∞, and the factor $[\cos\lambda + 1 - z^2/2]^{1/2}$ by its values $(2\cos\lambda)^{1/2}$ at the lower limit. The last expression is item 11 of Tables I, p. 138. Therefore, with $z = 2\,\mathrm{Pe} \sin\lambda/2$,

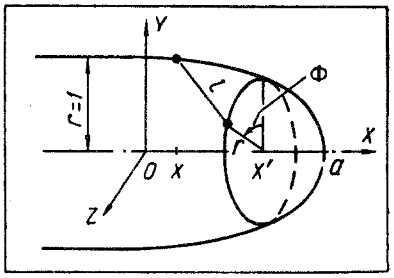

Fig. 7. Cylindrical dendrite with spheroidal cap.

$$\mathrm{Pe} \gg 1: \ u_j \simeq \frac{\mathrm{Pe}}{\pi} \int\limits_0^{\pi/2} \sin\lambda \sqrt{\cos\lambda} \times$$

$$\times \exp(\mathrm{Pe}\sin\lambda) K_0(2\,\mathrm{Pe}\sin\lambda/2)\, d\lambda. \tag{72}$$

Note that at Pe $\lambda \gg 1$, the integrand is exponentially small. Numerical integration of this expression for Pe = 10, 10^2, 10^4 yields the function plotted in Fig. 2.

Among the axial points the tip $(a, 0, 0)$ the origin $(0, 0, 0)$ and the far-away points $|x| \gg 1$ are again of principal interest. Writing

$$t = l + a - x', \quad x' = -ta^2 + a[t^2(a^2-1) + 2ta + 1]^{1/2}$$

$$\frac{dx'}{l} = -\frac{a\,dt}{[t^2(a^2-1) + 2ta + 1]^{1/2}}, \tag{73}$$

(65) becomes, for the tip

$$u_t = 1 - \exp\{-\mathrm{Pe}[a + (a^2+1)]^{1/2}\} -$$

$$- a\,\mathrm{Pe} \int\limits_0^{a+(a^2+1)^{1/2}} [t^2(a^2-1) + 2ta + 1]^{-1/2}\, t \exp(-\mathrm{Pe}\, t)\, dt. \tag{74}$$

For the special case of a spherical cap, $a = 1$, this becomes

$$u_t = 1 - \exp[-\mathrm{Pe}(1+\sqrt{2})] - \mathrm{Pe} \times \int\limits_1^{(3+2^{3/2})^{1/2}} \frac{s^2-1}{2} \exp[-\mathrm{Pe}(s^2-1)/2]\, ds =$$

$$= \frac{1}{2} + \frac{1}{2}[(3+2^{3/2})^{1/2} - 2] \exp[-(1+\sqrt{2})\mathrm{Pe}] + \frac{1}{2}\sqrt{\frac{\pi\,\mathrm{Pe}}{2}} \left(1 - \frac{1}{\mathrm{Pe}}\right) \times$$

$$\times \exp(\mathrm{Pe}/2) \left\{ \mathrm{erf}\left[\left(\frac{3}{2} + 2^{3/2}\right)\mathrm{Pe}\right]^{1/2} - \mathrm{erf}(\mathrm{Pe}/2)^{1/2} \right\} = \begin{cases} 1 - \dfrac{1}{\mathrm{Pe}} + \dfrac{2}{\mathrm{Pe}^2} - \ldots, & \mathrm{Pe} \gg 1, \\ 0.942\,\mathrm{Pe} - 0.089\,\mathrm{Pe}^2 \ldots, & \mathrm{Pe} \ll 1. \end{cases} \tag{75}$$

For the oblate spheroid, $a < 1$, we write $\tau = t - a/(1-a^2)$, and (74) reduces to

$$u_t = 1 - \exp\left\{-\mathrm{Pe}[a + (1+a^2)^{1/2}] - \frac{a\,\mathrm{Pe}}{\sqrt{1-a^2}} \exp\left(-\frac{a\,\mathrm{Pe}}{1-a^2}\right) \times\right. \tag{76}$$

$$\times \int_{-a/(1-a^2)}^{\sqrt{a^2+1}-a^3/(1-a^2)} \frac{\tau + a/(1-a^2)}{[(1-a^2)^{-2} - \tau^2]^{1/2}} \exp(-\mathrm{Pe}\,\tau)\,d\tau =$$

$$= \begin{cases} 1 - \dfrac{a}{\mathrm{Pe}} + \dfrac{2a^2}{\mathrm{Pe}^2} - \dfrac{9a^3 + 3a(1-a^2)}{\mathrm{Pe}^3} + \ldots, \quad \mathrm{Pe} \gg 1, \\ \mathrm{Pe}\left[\sqrt{a^2+1} + a^2 \dfrac{g_1 - a}{1-a^2}\right] - \mathrm{Pe}^2\left[\dfrac{g_1^2}{2} + \right. \\ \left. + \dfrac{a^2\{g_1(2a - g_2) - 3/2\}}{(1-a^2)^2}\right] + \ldots - \dfrac{a^2\,\mathrm{Pe}}{(1-a^2)^{3/2}} \times \\ \times [\arcsin a - \arcsin g_2]\left[1 - \dfrac{(1+2a)\,\mathrm{Pe}}{2(1-a^2)} + \ldots\right], \quad \mathrm{Pe} \ll 1, \end{cases}$$

$$g_1(a) \equiv [2a^2 + 1 + 2a\sqrt{1+a^2}]^{1/2}, \quad g_2(a) = a^3 + (a^2 - 1)\cdot\sqrt{1+a^2}.$$

For the prolate spheroid, $a > 1$, we write $\tau = t + a/(a^2-1)$ and formula (74) reduces to

$$u_t = 1 - \exp\{-\mathrm{Pe}[a + \sqrt{a^2+1}] - \frac{a\,\mathrm{Pe}}{\sqrt{a^2-1}} \exp\left(\frac{a\,\mathrm{Pe}}{a^2-1}\right) \times$$

$$\times \int_{a/(a^2-1)}^{\sqrt{a^2+1}+a^3/(a^2-1)} [\tau^2 - 1/(1-a^2)^2]^{-1/2} [\tau - a/(a^2-1)] \exp(-\mathrm{Pe}\,\tau)\,d\tau =$$

$$= \begin{cases} 1 - \dfrac{a}{\mathrm{Pe}} + 2\dfrac{a^2}{\mathrm{Pe}^2} - \ldots, \quad \mathrm{Pe} \gg 1, \\ \mathrm{Pe}\left[a + \sqrt{a^2+1} + \dfrac{a(1-ag_1)}{a^2-1} + \dfrac{a^2}{(a^2-1)^{3/2}} \times \right. \\ \left. \times \ln \dfrac{g_2 + a\sqrt{a^2-1}\,g_1}{a + \sqrt{a^2-1}}\right] - \mathrm{Pe}^2\left[\dfrac{g_1^2}{2} + \dfrac{a^2}{(a^2-1)^2} \times \right. \\ \left. \times \left(2ag - \dfrac{3}{2} - \dfrac{g_1 g_2}{2}\right) - \dfrac{a(a^2 + 1/2)}{(a^2-1)^{5/2}} \ln \dfrac{g_2 + a\sqrt{a^2-1}\,g_1}{a + \sqrt{a^2-1}}\right] + \ldots, \\ \mathrm{Pe} \ll 1. \end{cases} \qquad (77)$$

At the origin (x, y, z) = (0, 0, 0) we obtain, with

$$t = l - x', \quad x' = -ta^2 + a[t^2(a^2-1) + 1]^{1/2}, \quad \frac{dx'}{l} = -\frac{a\,dt}{[t^2(a^2-1)+1]^{1/2}}, \qquad (78)$$

the results

$$a = 1: \quad u_{0r} = 1 - \frac{1 - \exp(-\mathrm{Pe})}{\mathrm{Pe}}, \qquad (79)$$

$$a < 1: \quad u_{0r} = \begin{cases} 1 - \dfrac{a}{\mathrm{Pe}}\left[1 + \dfrac{3(1-a^2)}{\mathrm{Pe}^2} + \ldots\right], \quad \mathrm{Pe} \gg 1, \\ \dfrac{\mathrm{Pe}}{1+a}\left[1 + \dfrac{\mathrm{Pe}\,a}{2(1-a)}\left\{\dfrac{\arcsin(1-a^2)}{\sqrt{1-a^2}} - a\right\} + \ldots\right], \quad \mathrm{Pe} \ll 1, \end{cases} \qquad (80a)$$

$$a > 1: \quad u_{0r} = \begin{cases} 1 - \dfrac{a}{\mathrm{Pe}}\left[1 - \dfrac{3(a^2-1)}{\mathrm{Pe}^2} + \ldots\right], \quad \mathrm{Pe} \gg 1, \\ \dfrac{\mathrm{Pe}}{1+\mathrm{Pe}}\left[1 + \dfrac{\mathrm{Pe}\,a}{2(a-1)}\left\{\dfrac{\ln(a + \sqrt{a^2-1})}{\sqrt{a^2-1}} + a\right\} + \ldots\right], \\ \mathrm{Pe} \ll 1. \end{cases} \qquad (80b)$$

Finally, for $|x| \gg 1$, noting

$$l = |x|\left[1 - \frac{x'}{x} + \frac{1}{2}\,\frac{1 - x'^2/a^2}{x^2} - \dots\right], \tag{81}$$

we obtain

$$\frac{a^2 u}{\mathrm{Pe}}\,|x| \simeq \begin{cases} \int\limits_0^a \exp\left[-2\,\mathrm{Pe}\,(x - x')\right]\left[x' + \dfrac{x'^2}{x}\right]dx', & x > 0, \\ \int\limits_0^a \exp\left[-(\mathrm{Pe}/2|x|)(1 - x'^2/a^2)\right]\left[x' - \dfrac{x'^2}{|x|}\right]dx', & x < 0, \end{cases} \tag{82}$$

and find

$$x \gg 1:\; u \simeq \frac{\exp(-2\,\mathrm{Pe}\,x)}{2a^2 x}\left\{\left[a - \frac{1}{2\mathrm{Pe}} + \frac{1}{x}\times\left(a^2 - \frac{a}{\mathrm{Pe}} + \frac{1}{2\mathrm{Pe}^2}\right)\right]\exp(2\,\mathrm{Pe}\,a) + \right.$$

$$\left. + \frac{1}{2\mathrm{Pe}} - \frac{1}{2\mathrm{Pe}^2 x}\right\}, \tag{83a}$$

$$x \ll -1:\; u \simeq 1 - \exp(-\lambda) - \frac{a\,\mathrm{Pe}}{x^2}\exp(-\lambda)\times\left(\frac{1}{3} + \frac{\lambda}{5} + \frac{\lambda^2}{7\cdot 2!} + \frac{\lambda^3}{9\cdot 3!} + \dots\right),\quad \lambda \equiv \frac{\mathrm{Pe}}{2|x|}. \tag{83b}$$

In (83b) we have made the additional assumption that $\mathrm{Pe}/|x|^3$ is negligible.

VI. Summary

One of the objectives of the study was to exhibit the use of traveling heat source method in solution of dendrite growth problems. Another was to elucidate Fisher's estimate of the temperature of a dendrite with hemispherical head and cylindrical tail. Plausibility arguments suggest that, at low velocities, the heating up of the dendrite should be proportional to the speed of growth; Fisher specified the numerical constant, on the basis of back-of-an-envelope calculation, in the formula

$$(c/\lambda)\,(T_{tip} - T_\infty) = u_t = \mathrm{Pe} = RV/2\kappa$$

The rigorous results, established in this report, barely modifies the value to 0.94 Pe. It was also shown that, at very high speeds, u_t approaches 1. In addition, some further particulars of the temperature field were determined both at small velocities ($\mathrm{Pe} \ll 1$) and at large velocities ($\mathrm{Pe} \gg 1$), and in some instances also at intermediate velocities; for both two- and three-dimensional geometries, not only for circular heads, Figs. 2 and 4, but also for elliptical heads, Figs. 5 and 6.

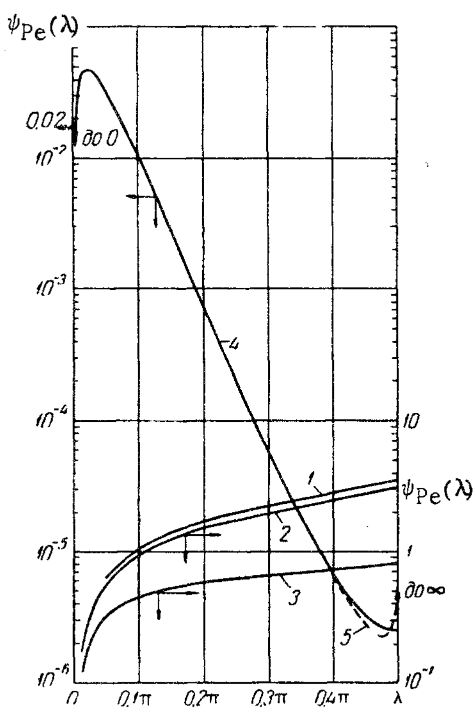

Fig. 8. The auxiliary function $\psi_{\mathrm{Pe}}(\lambda)$.

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